Berry Phase & Chern Numbers

Daniel Mandragona

Texas A&M University

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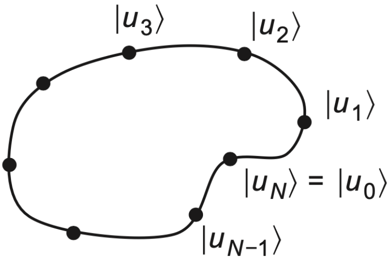
- *Berry Phase* is the total amount of phase accumulated by the loop.

Berry Phase (Discrete Formulation)

Consider *n*-many unit quantum states arranged in a loop as such:



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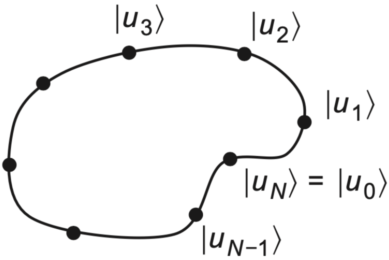

Berry Phase

We may define the *Berry Phase* as:

$$BP := -arg\left[\langle u_0|u_1\rangle \langle u_1|u_2\rangle \cdots \langle u_{N-1}|u_0\rangle\right] \tag{1}$$

Gauge Transformations

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- The numbers will be different, yet nothing changed about your *driving experience*.

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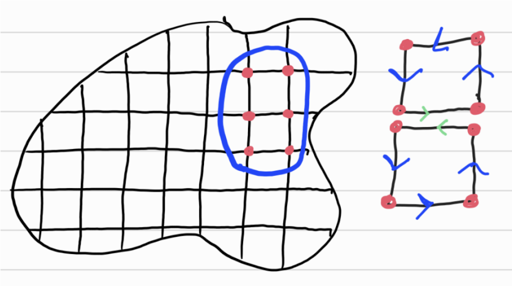
B.P. Gauge Independence

Applying the gauge transformation to every state:

$$
\begin{aligned}
BP &:= -arg\left[\langle u_0|u_1\rangle \langle u_1|u_2\rangle \cdots \langle u_{n-1}|u_0\rangle\right] \\
&\mapsto -arg\left[(e^{i(-\beta_0+\beta_1-\beta_1+\beta_2-\beta_2+\cdots+\beta_0)} \langle u_0|u_1\rangle \langle u_1|u_2\rangle \cdots \langle u_{n-1}|u_0\rangle\right] \\
&= -arg\left[e^0 \langle u_0|u_1\rangle \langle u_1|u_2\rangle \cdots \langle u_{n-1}|u_0\rangle\right] \\
&= BP
\end{aligned}
$$

Berry Flux

- Consider now a 2D lattice of these states:



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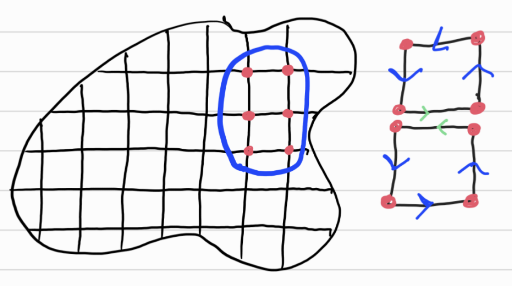

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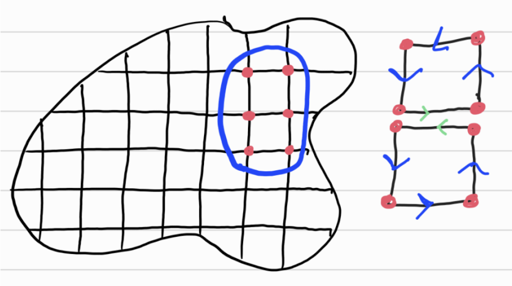

- Let's calculate the Berry Phase of the loop along the boundary.
- Alt. Approach: Add up BP of the enclosed plaquettes.
  - Each internal edges is included twice with opposite orientations.
  - The BP for a plaquette, $\square$, is called *Berry Flux*, denoted as $F_{\square}$.

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- This number Q is the *Chern Number*.

Continuous Formulation

- $ln \langle u_\lambda | u_{\lambda+d\lambda} \rangle = ln \langle u_\lambda | \Big(|u_\lambda\rangle + d\lambda \, |\partial_\lambda u_\lambda\rangle + \dots \Big) \approx d\lambda \, \langle u_\lambda | \partial_\lambda u_\lambda \rangle$
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- The integrand is purely imaginary $\implies \gamma_L = \oint_L i \langle u_\lambda | \partial_\lambda u_\lambda \rangle \, d\lambda$

$$\begin{aligned} 2Re \langle u_\lambda | \partial_\lambda u_\lambda \rangle &= \langle u_\lambda | \partial_\lambda u_\lambda \rangle + \overline{\langle u_\lambda | \partial_\lambda u_\lambda \rangle} \\ &= \langle u_\lambda | \partial_\lambda u_\lambda \rangle + \langle \partial_\lambda u_\lambda | u_\lambda \rangle \\ &= \partial_\lambda \langle u_\lambda | u_\lambda \rangle = 0 \end{aligned}$$

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- Apply Stokes' Theorem (to what you can):

$$\oint_L \vec{A}(\lambda) \cdot d\lambda + 2\pi Q = \iint_S \Omega(\lambda) dS$$

Chern Theorem

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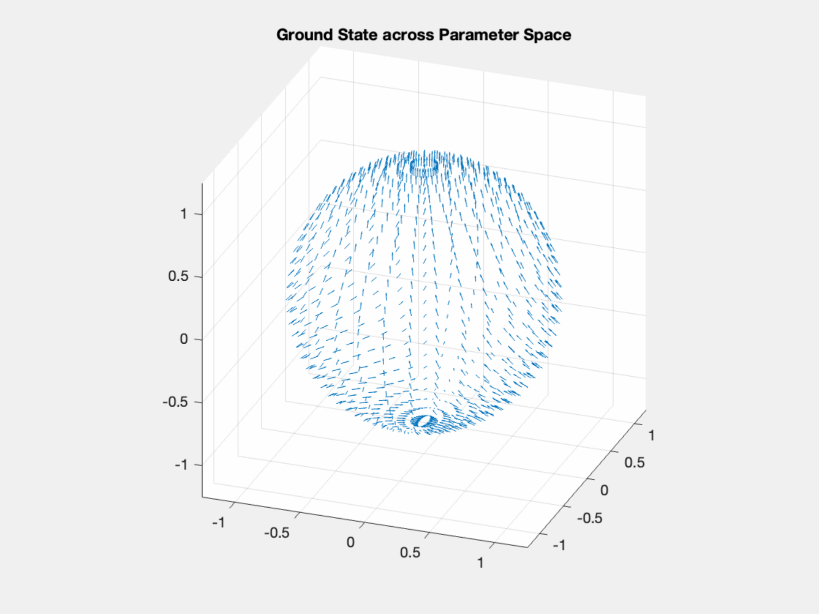
For a closed 2D manifold we have that the integral of the Berry Curvature is $2\pi Q$, where $Q \in \mathbb{Z}$ is the *Chern Number*.

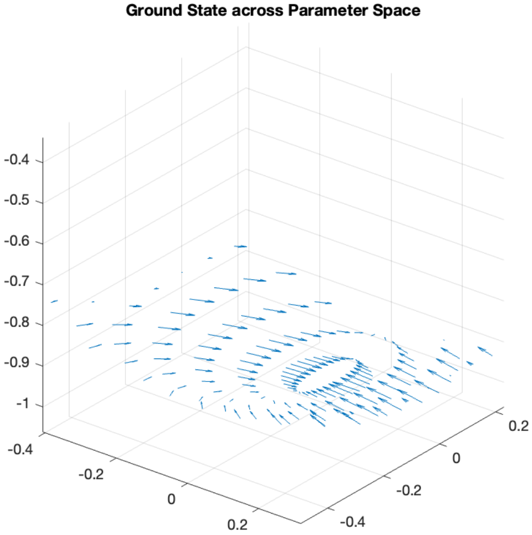
$$\iint_S \Omega(\lambda) \cdot d\mathbf{S} = 2\pi Q \tag{3}$$

"Note that when the Chern number is nonzero, it is impossible to construct a smooth and continuous gauge over the entire surface S" [Vanderbilt].

Gauge Obstruction (1)

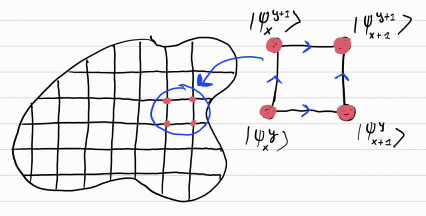


Gauge Obstruction (2)



Calculating Chern Number for Torus (Fukui et al.)

- Step 1: Discretize the Torus into plaquettes.



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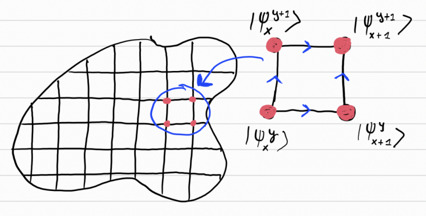

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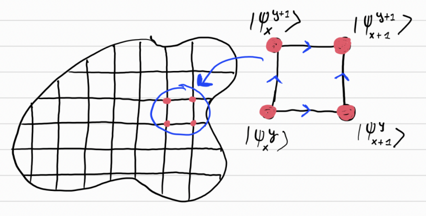

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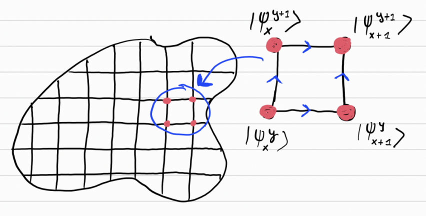

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- Step 4: Calculate Berry Flux for each plaquette and sum:

$$\tilde{F}_x^y = ln\Big(U(e_1)U(e_2)U(e_3)^{-1}U(e_4)^{-1}\Big) \quad (4)$$

Application (1)

- We follow Fukui et al. and apply the method to the following family of Hamiltonians parameterized by $\vec{\mathbf{k}} = (k_x, k_y)$:

$$H(\vec{\mathbf{k}}) = \begin{pmatrix} -2t\cos(k_y - \frac{2}{3}\pi) & -t & -te^{-3ik_x} \\ -t & -2t\cos(k_y - \frac{4}{3}\pi) & -t \\ -te^{3ik_x} & -t & -2t\cos(k_y - 2\pi) \end{pmatrix}$$

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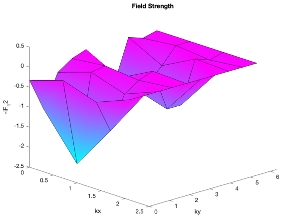
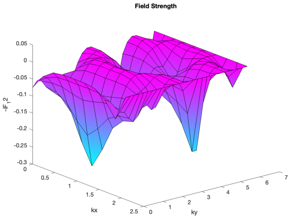
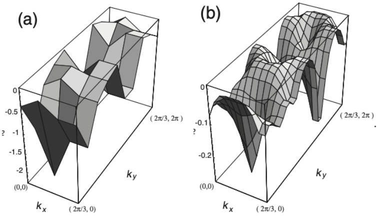
- This family of matrices is obtained from the floquet representation of the Hamiltonian describing spinless fermions subjected to an eternal magnetic field undergoing certain flux constraints.

Results

We reproduced the results of Fukui et al., i.e. $Q = -2$ and found similar $\tilde{F}$-surfaces:







Application (2)

- Qi-Wu-Zhang Model is given by a Hamiltonian in k-space with an additional parameter, u:

$$H(\vec{\mathbf{k}}, u) = \begin{pmatrix} u + \cos(k_x) + \cos(k_y) & \sin(k_x) - i\sin(k_y) \\ \sin(k_x) + i\sin(k_y) & -u - \cos(k_x) - \cos(k_y) \end{pmatrix}$$

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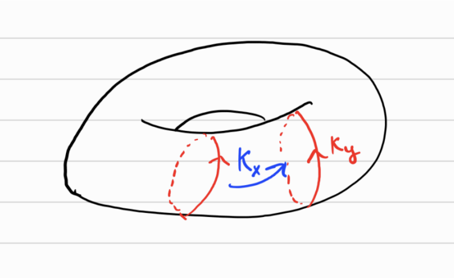
- The Chern Number as a function of u for this model is:

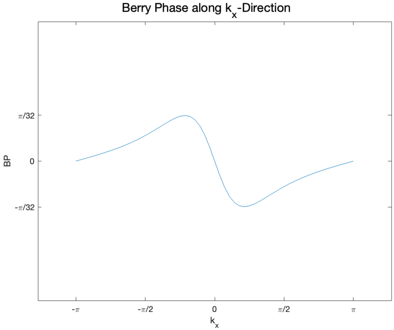
$$Q(u) = \begin{cases} 0 & \text{if } |u| > 2 \\ -1 & \text{if } -2 < u < 0 \\ +1 & \text{if } 0 < u < 2 \end{cases}$$

Which is exactly what we see in our code.

Visualizing QWZ Model's Chern Number



(a) Brillouin Zone



(b) BP along rings of the torus.

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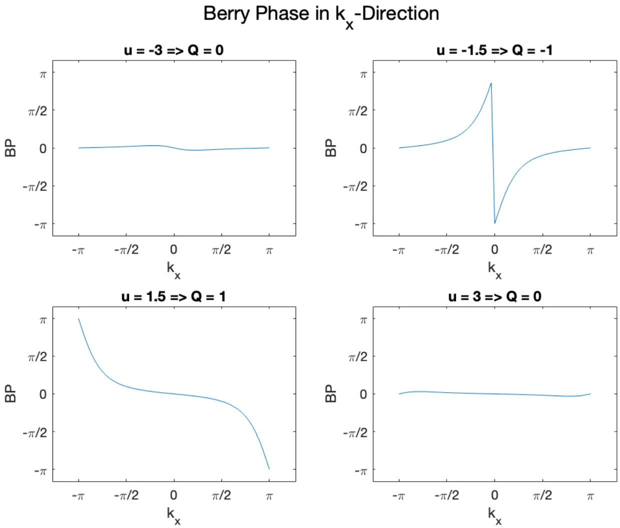

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