QEC: Classical Errors to the Surface Code

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Introduction

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- However, QCs are incredibly fragile and susceptible to noise.
  - The *basic* theory assumes a closed quantum system.
- This leads to a paradox:

How can a completely closed quantum system be manipulated by an observer?

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    - There's no copying a quantum state.

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- The situation is much better than in the quantum setting, however.
  - Error rates are extremely low (e.g., one per billion operations).
  - Strong error correction methods, like the Hamming code are still in use (e.g., in RAM) today.

The Repetition Code: Encoding

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  - $0 \to 000$ (Logical 0)
  - $1 \to 111$ (Logical 1)
- The encoded blocks (000, 111) are called **codewords**.

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- This simple code can correct any *single* bit-flip error.

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- We can **improve** the code by using longer odd-length repetitions.

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  - **Parity Check Matrix (H):** Detects errors.

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- Example: The message 1011 is encoded as:

$$(1, 0, 1, 1)G = (1, 0, 1, 1, 0, 1, 0)$$

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- If the result of Hc^T is non-zero, an error has been detected.

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- For a single-bit error at position i, the syndrome s will be equal to the i-th column of H.
- This allows us to identify and correct the single-bit error.

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- A code with distance d can:
  - Detect up to $d-1$ errors.
  - Correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors.
- **Benefit:** More efficient than the repetition code (7 bits for 4 logical bits vs. 3 for 1).

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Why Quantum Error Correction is Harder

QEC is more complex than classical error correction for several reasons:

1. **More error types:** Beyond bit-flips, qubits can have phase-flips, and a continuous range of other errors.
2. **No-Cloning Theorem:** We cannot simply copy a qubit to create redundancy.
3. **Measurement is destructive:** Measuring a qubit to check for errors collapses its state, destroying the information we want to protect.

Seems difficult...

The 3-Qubit Bit-Flip Code: Encoding

A quantum version of the repetition code to correct bit-flip errors.

- Goal: Encode a single logical qubit:

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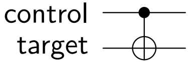
- This is NOT cloning. The No-Cloning Theorem forbids making independent copies like:

$$|\psi_{\text{copied}}\rangle = (\alpha|0\rangle + \beta|1\rangle)^{\otimes 3}$$

- The Controlled-NOT (CNOT) gate is a two-qubit gate that flips the target qubit if the control qubit is $|1\rangle$.



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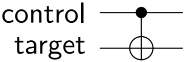
- Matrix representation:

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

acting on bases: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.



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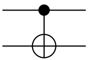
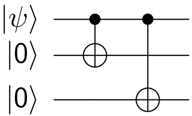
- Circuit diagram:

control
target



Encoding Circuit for 3 Qubit Bit-Flip Code

We use two CNOT gates to create the logical state $|\psi_L\rangle$:



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix}$$

Example (Creating $|\psi_L\rangle$)

The circuit transforms the initial state $|\psi\rangle|0\rangle|0\rangle$:

$$\begin{aligned} |\psi_{initial}\rangle &= (\alpha|0\rangle + \beta|1\rangle)|00\rangle = \alpha|000\rangle + \beta|100\rangle \\ &\xrightarrow{CNOT_{12}} \alpha|000\rangle + \beta|110\rangle \\ &\xrightarrow{CNOT_{13}} \alpha|000\rangle + \beta|111\rangle = |\psi_L\rangle \end{aligned}$$

Quantum Errors: Pauli Operators

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It flips the state: $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$.

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- These operators satisfy the property: $XYZ = i$, so $Y = iXZ$, and are *their own inverses*.

Detecting Errors Without Measurement

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- For the bit-flip code, the stabilizers are $Z_1 Z_2$ and $Z_2 Z_3$.
- We measure a *joint* property of these operators, not the individual qubits. We collapse the *correlation*, not the individual states.

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- Your quantum state can then be *written* w.r.t. the bases of each one of the Projection Matrices.
- Measurement collapses the state onto the eigenspace associated with the measured eigenvalue. The probability of collapsing to a specific eigenspace k is $p(k) = \|P_k|\psi\rangle\|^2$.".

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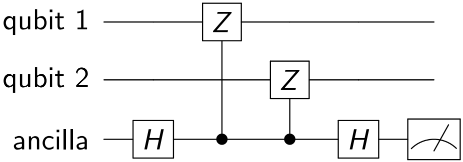

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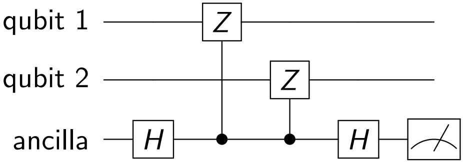

- The measurement outcome of the ancilla tells us the eigenvalue of $Z_1 Z_2$.

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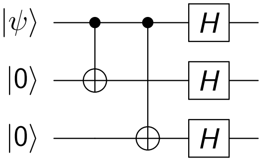
- **Encoding**:

$$|\psi_L\rangle = \alpha|+++\rangle + \beta|---\rangle$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Phase-Flip Code: Encoding Circuit

The encoding circuit uses Hadamards to change basis.



- The stabilizers for this code are X_1X_2 and X_2X_3.
- A Z error on one qubit will flip the sign of one or both stabilizer measurements, revealing the error.

Phase-Flip Code: Syndrome Measurement

The syndrome measurement circuit is analogous to the bit-flip code's, but with controlled-X gates.

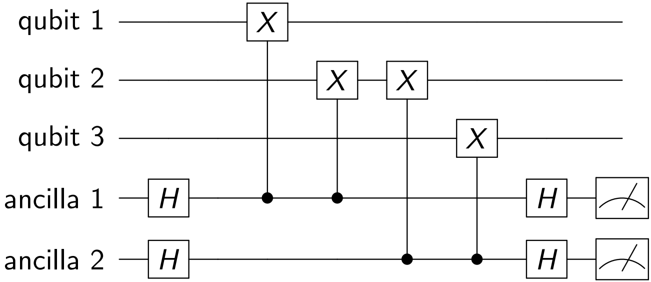


Figure: Circuit for measuring errors in the phase-flip code.

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- It encodes 1 logical qubit into 9 physical qubits.

Shor Code: Encoding

Encoding is a two-step concatenation:

1. **Outer Code (Phase-Flip):** The logical qubit is first encoded to 3 qubits.
   - $\alpha|0\rangle + \beta|1\rangle \to \alpha|+++\rangle + \beta|---\rangle$
2. **Inner Code (Bit-Flip):** Each of these 3 qubits is then encoded into 3 more qubits.
   - $|+\rangle \to \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
   - $|-\rangle \to \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$

This results in a 9-qubit state:

$$|0_L\rangle \to \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$$
$$|1_L\rangle \to \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)$$

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The 9 qubits are grouped into three blocks of three. Each block is a bit-flip code.

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Requirements for Error Discretization

- Stabilizers must be able to actually detect the Pauli error basis.
- *Simultaneous* stabilizer measurements require their stabilizers to **commute**.

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- **Conclusion:** By correcting only the discrete Pauli errors, we can correct any arbitrary single-qubit error.

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- Caution: We can't include $-I$ in $\mathcal{S}$ (next slide will explain why).

How Stabilizer Codes Work

- The **codespace** is the subspace of states $|\psi\rangle$ that are left unchanged ($+1$ eigenvalue) by all stabilizers:

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- We measure all stabilizer generators to get the **error syndromes**.

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- Caveat: This requires a non-degenerate quantum code, meaning that every correctable error has a unique syndrome.

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- The four commuting stabilizer generators are:

$$S_1 = X \otimes Z \otimes Z \otimes X \otimes I$$
$$S_2 = I \otimes X \otimes Z \otimes Z \otimes X$$
$$S_3 = X \otimes I \otimes X \otimes Z \otimes Z$$
$$S_4 = Z \otimes X \otimes I \otimes X \otimes Z$$

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- The logical operators for the 5-qubit code are:

$$\bar{X} = X \otimes X \otimes X \otimes X \otimes X$$
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- The code distance d is the minimum weight (single qubit Pauli operations) of a non-trivial logical operator.
- For the 5-qubit code, $d = 3$. This means it can correct $t = \lfloor \frac{3-1}{2} \rfloor = 1$ error.

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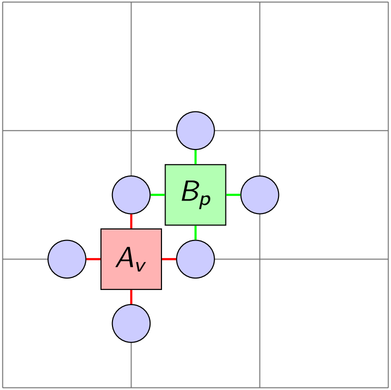
- It is a **topological code**: its properties are protected by the global structure of the system.

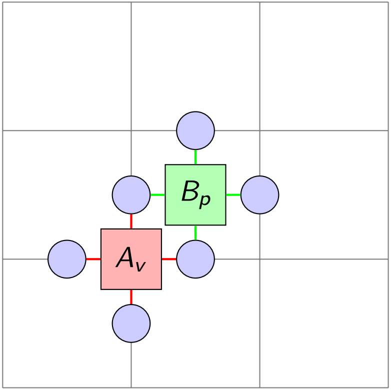
Surface Code: The Lattice

- Data qubits reside on the **edges** of a 2D lattice.



Surface Code: The Lattice

- Data qubits reside on the **edges** of a 2D lattice.
- Stabilizers are defined based on the vertices and faces (plaquettes) of the lattice.



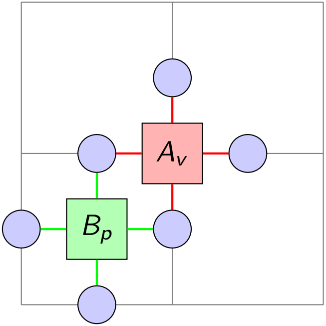
Surface Code: The Stabilizers

- **Star operators (A_v):** Product of X on qubits meeting at a vertex v.

$$A_v = \bigotimes_{i \in \mathrm{star}(v)} X_i$$

- **Plaquette operators (B_p):** Product of Z on qubits bounding a face p.

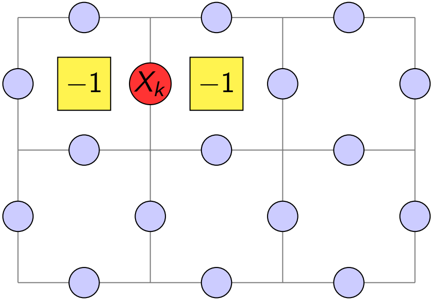
$$B_p = \bigotimes_{i \in \mathrm{boundary}(p)} Z_i$$



Surface Code: Error Detection

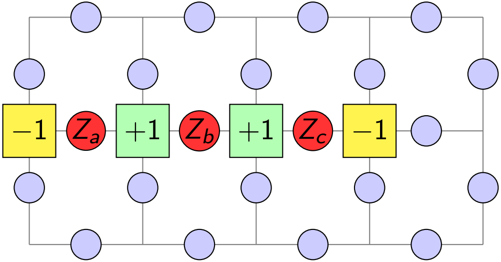
Errors create pairs of **defects** at the ends of error chains.

- X **errors:** Anti-commutes with two adjacent **plaquette** operators, creating a pair of Z-defects.
- Z **errors:** Anti-commutes with two adjacent **star** operators, creating a pair of X-defects.



Error Chains

- A string of errors of the same type creates defects only at the **endpoints** of the chain.
- Example: A chain of three Z errors (Z_a, Z_b, Z_c).

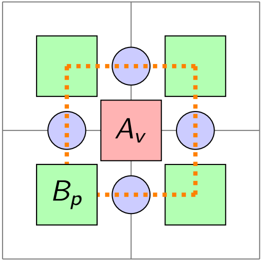


Do X Errors Form Chains?

- Motivation: Want to show[1] that X errors also form chains.

[1] Thank you PanQEC (Eric Huang & Arthur Pesah)!

- The **Dual Lattice** is created by rotating all of the qubit edges by 90 degrees (orange dotted lines).



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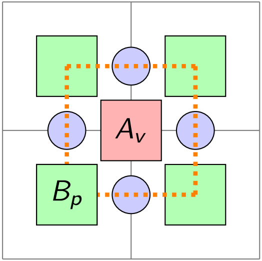

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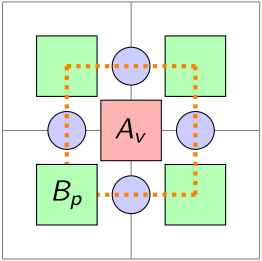
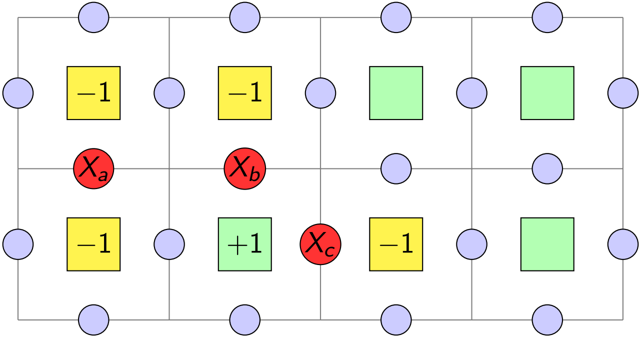
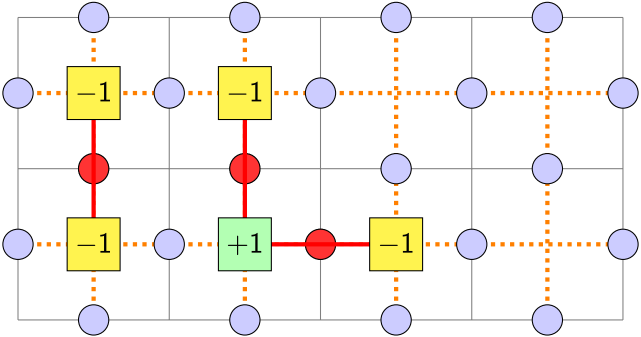


- **Result**: Star Operators $\Leftrightarrow$ Plaquette Operators.
- An X error chain on the dual lattice behaves like a Z error chain on the primal lattice.

X Error Chain on the Primal Lattice



X Error Chain on the Dual Lattice



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    - If the loop is a stabilizer, then $E_{guess}E_{true}|\psi_L\rangle = |\psi_L\rangle$.
    - The correction works!

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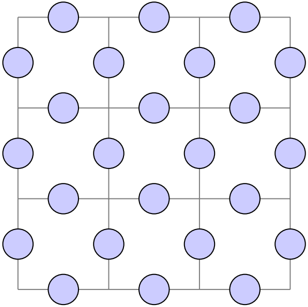

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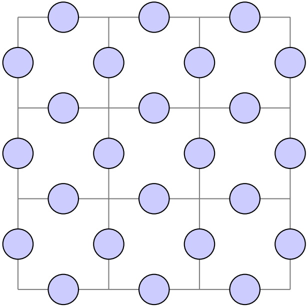

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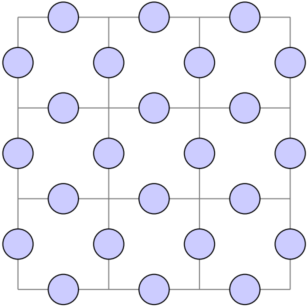
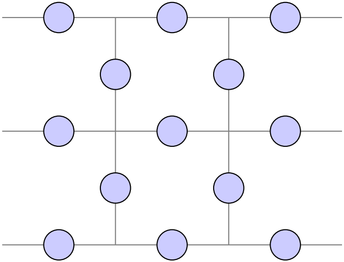


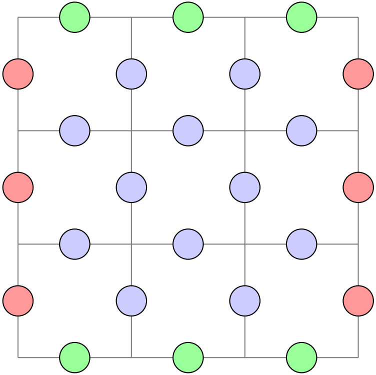
- To get a logical operator we need to create a non-trivial non-detectable operation.
- We can move the endpoints into each other, but this just creates stabilizer given the topology:

Different Topologies

- To have logical operations we need a different topology:



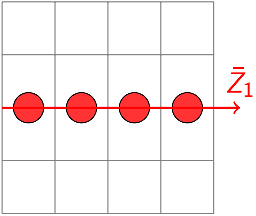
3x3 Planar Code

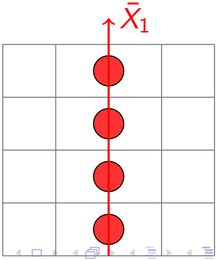


$d = 3$ Toric Code

Logical Operators on the Torus

- On a torus, there are non-trivial loops that cannot be shrunk to a point.





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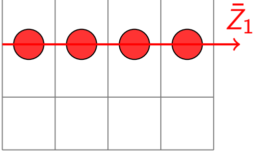
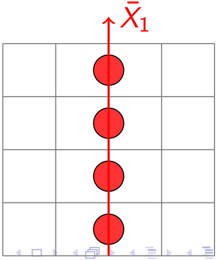
- These correspond to logical operators, the Toric Code has four of them.





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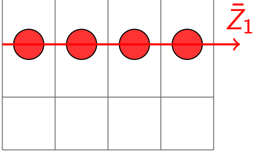

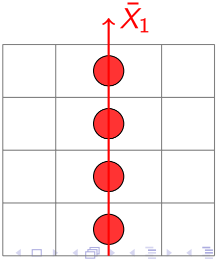

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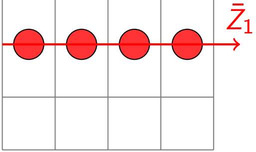
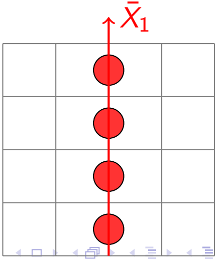
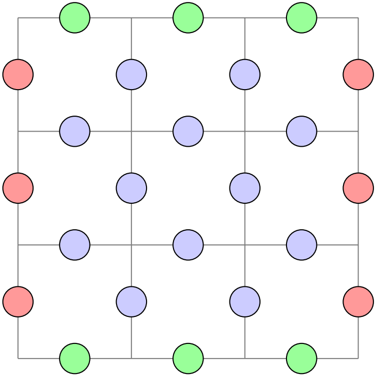




Toric Code Properties

- For an $L \times L$ toric code, we have $2L^2$ data qubits.



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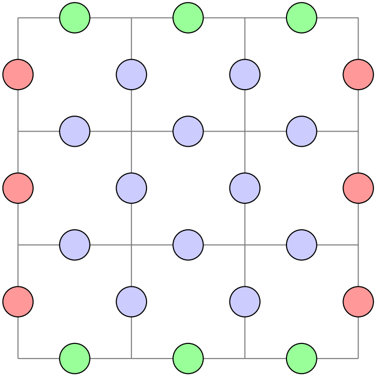

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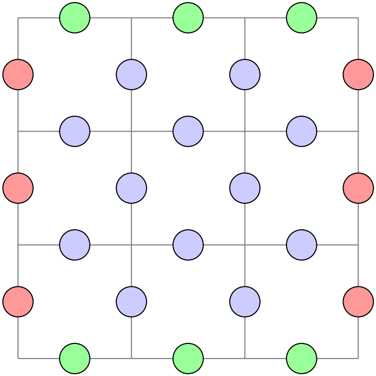

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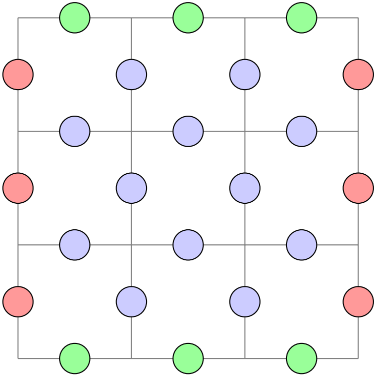

3×3 Toric Code

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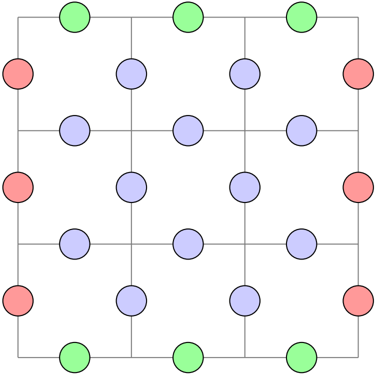

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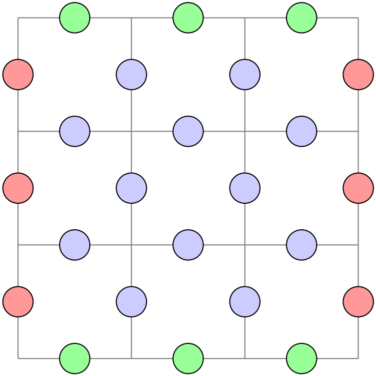
- So, the toric code is a $[[2L^2, 2, L]]$ code; correcting up to $t = \lfloor (L-1)/2 \rfloor$ errors.



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