WEYL QUANTIZATION

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ABSTRACT

We shall describe the method of Weyl Quantization outlined by Brian Hall in [1] which acts as an operator on the space of L^2 functions over the classical phase space i.e., $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, onto the space of Hilbert-Schmidt operators over $L^2(\mathbb{R}^n)$. We will then show some nice properties of this operator.

Contents

1 What is Quantization?

Quantum mechanics states that for every real-valued function, f, on the classical phase space there is an associated self-adjoint operator, $\hat{f}$, on the quantum Hilbert space. We shall call $\hat{f}$ the **quantization** of f.

Example 1 (Need for Quantization)**.** *Consider the classical observables x and p. Then since the commutator of their quantizations: $[X, P] = i\hbar$, we know that $(XP)^* = PX \neq XP$. So the straight forward replacement will not lend us a quantization of the classical functions xp. Instead we could try something like:*

$$\hat{x}\hat{p} = \frac{1}{2}(XP + PX)$$

2 Weyl Quantization of Polynomial Observables

For simplicity we limit ourselves to systems with one degree of freedom, and classical observables that are polynomials in x and p. We also note that operators that we will be considering may fail to be essentially self-adjoint even if they are symmetric such as the operator:

$$P^2 - cX^4 \qquad (c > 0)$$

which is not essentially self-adjoint on $C_c^\infty(\mathbb{R})$.

We see that the trouble of quantization comes from choosing an ordering of X and P. We will see that **Weyl Quantization** effectively considers all possible orderings and weighs them evenly:

Example 2 (Weyl Quantization of x^2p^2)**.** *This would give us:*

$$\frac{1}{6}\Big(X^2P^2 + XPXP + PXPX + P^2X^2 + PXXP + XP^2X\Big)$$

We may therefore define the *Weyl Quantization* scheme as:

$$Q_{weyl}(x^jp^k) = \frac{1}{(j+k)!}\sum_{\sigma\in S_{j+k}} \sigma(X,X,\dots,X,P,P,\dots,P) \tag{1}$$

It is clear from this definition that the Weyl operator is linear. Furthermore we have the following theorem:

Theorem 1. *We see that the Weyl Quantization* uniquely *satisfies the following property:*

$$Q_{weyl}\big((ax+bp)^j\big) = (aX+bP)^j$$

Proof. Let (a_i) and (b_i) be sequences in $\mathbb{C}$. Then we can see that:

$$\begin{aligned} &Q_{weyl}\big((a_1x+b_1p)\cdots(a_jx+b_jp)\big)\\ &\quad= \frac{1}{j!}\sum_{\sigma\in S_j}\sigma(a_1X+b_1P,\dots,a_jX+b_jP)\end{aligned}$$

This is because of linearity of the Weyl operator and since $\sigma(X+P,X+P) = \sigma(X,X)+\sigma(P,X)+\sigma(X,P)+\sigma(P,P)$.

Now taking the case where $a_i = a$ and $b_i = b$ we get that each selection of aX or bP is indistinguishable and each permutation is the same object. Thus, since $|S_j| = j!$ we cancel out the $\frac{1}{j!}$ in front of the sum and are left with:

$$(aX+bP)^j$$

as desired. Conversely, if we say that Q is a linear map of the space of polynomials into the space of operators that satisfies the theorem property, then we may denote V_j as the space of homogeneous polynomials of degree j (every nonzero term has variables whose degrees sum to the same value as every other term) where Q agrees with Q_{weyl}. By assumption we have that V_j contains all polynomials of the form $(ax+bp)^j$, and by appendix [6] we know that this gives us all of V_j, hence $Q = Q_{weyl}$. □

3 Weyl Quantization in $\mathbb{R}^{2n}$

Instead of considering Weyl Quantization for polynomials in $\mathbb{R}^2$ as we did above, we can generalize to $\mathbb{R}^{2n}$ with the formula:

$$Q_{weyl}\big((\mathbf{a}\cdot\mathbf{x}+\mathbf{b}\cdot\mathbf{p})^j\big) = \big(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P}\big)^j \tag{2}$$

We can also see that (2) will interact nicely with multiplication by a factor of $\frac{i^j}{j!}$ and also summation. Hence we can extend to complex exponentials to get:

$$Q_{weyl}\big(e^{i(\mathbf{a}\cdot\mathbf{x}+\mathbf{b}\cdot\mathbf{p})}\big) = e^{i(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P})}$$

With complex exponentials under our belt, we may quantize functions using the Fourier transform. So if we have some function f on the classical phase space such that it may be expressed via the Fourier transform:

$$f(\mathbf{x},\mathbf{p}) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}}\hat{f}(\mathbf{a},\mathbf{b})e^{i(\mathbf{a}\cdot\mathbf{x}+\mathbf{b}\cdot\mathbf{p})}d\mathbf{a}d\mathbf{b}$$

we may quantize it as:

$$Q_{weyl}(f) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}}\hat{f}(\mathbf{a},\mathbf{b})e^{i(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P})}d\mathbf{a}d\mathbf{b} \tag{3}$$

We wish to now compute exactly what $\exp\big(i(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P})\big)$ is. We will use a useful formula proved in Appendix [6]:

$$e^{A+B} = e^{-\frac{[A,B]}{2}}e^Ae^B$$

which applies to bounded operators which also commute with their commutator ($[A,[A,B]] = [B,[A,B]] = 0$). If we take $A = i\mathbf{a}\cdot\mathbf{X}$ and $B = i\mathbf{b}\cdot\mathbf{P}$ (ignoring that these are unbounded operators, see Appendix [6] as to why they are unbounded), then we get:

$$e^{i(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P})} = e^{i\hbar\frac{\mathbf{a}\cdot\mathbf{b}}{2}} e^{i\mathbf{a}\cdot\mathbf{X}} e^{i\mathbf{b}\cdot\mathbf{P}}$$

We also know from Stone's Theorem that $(e^{i\mathbf{b}\cdot\mathbf{P}}\psi)(x) = \psi(\mathbf{x}+\hbar\mathbf{b})$. We therefore wish to prove:

$$(e^{i(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P})}\psi)(x) = e^{i\hbar\frac{\mathbf{a}\cdot\mathbf{b}}{2}} e^{i\mathbf{a}\cdot\mathbf{X}}\psi(\mathbf{x}+\hbar\mathbf{b}) \tag{4}$$

We can show this by replacing $\mathbf{a}$ and $\mathbf{b}$ with $t\mathbf{a}$ and $t\mathbf{b}$ on the right side of (4) and show that this gives us a strongly continuous one-parameter unitary group.

Theorem 2. *For all $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$, the operators $U_{\mathbf{a},\mathbf{b}}(t)\in L^2(\mathbb{R}^n)$ given by:*

$$\big(U_{\mathbf{a},\mathbf{b}}(t)\psi\big)(x) = e^{i\hbar t^2\frac{\mathbf{a}\cdot\mathbf{b}}{2}} e^{it\mathbf{a}\cdot\mathbf{X}}\psi(\mathbf{x}+\hbar t\mathbf{b}) \tag{5}$$

form a strongly-continuous one-parameter unitary group. Furthermore, its infinitesimal generator is:

$$\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P} \tag{6}$$

on $C_c^\infty(\mathbb{R}^n)$, and is essentially self-adjoint on this domain.

Proof. $U_{\mathbf{a},\mathbf{b}}(t)$ is just a rotation, hence it is unitary. It is also clear that this defines a group parametrized by t. Hence it is a unitary group. Furthermore, to show strong continuity we note that the operator is given by a product of two continuous functions multiplied by ψ. If we first start with ψ being continuous and compactly supported we will get that $U_{\mathbf{a},\mathbf{b}}(t)\psi$ is continuous at $t=0$, but this subspace is dense in $L^2(\mathbb{R}^n)$ and so we can extend to all of $L^2(\mathbb{R}^n)$.

We now check the formula for its infinitesimal generator. Taking the derivative at $t=0$ of the right side of (5) on the dense subspace of $L^2(\mathbb{R}^n)$, $C_c^\infty(\mathbb{R}^n)$ we get:

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} U_{\mathbf{a},\mathbf{b}}(t)\psi &= (i\mathbf{a}\cdot\mathbf{X})\psi(\mathbf{x}) + \psi'(\mathbf{x})(\hbar\mathbf{b})\\ &= i\mathbf{a}\cdot\mathbf{X}\psi - \frac{1}{i}\mathbf{b}\cdot\mathbf{P}\psi\end{aligned}$$

Now by using $\mathbf{P} = -i\hbar\frac{\partial}{\partial\mathbf{x}}$. Now, since the formula for the infinitesimal generator is $\frac{1}{i}$ multiplied by this derivative (given by Stone's Theorem), then we get exactly (6). Finally it can be shown that (6) is self-adjoint on $C_c^\infty(\mathbb{R}^n)$ (see [1] 9.40), and so we are done. □

We now wish to return to the formula of (3) where we quantize a function $f\in L^2(\mathbb{R}^n)$. Therefore Q_{weyl} is an operator on $L^2(\mathbb{R}^n)$ and we will show that it must be an integral operator for some respective kernel. If we assume that this must be the case then it is clear that the kernel of the integral of (3) using our new expression (4) will be:

$$e^{i\hbar\frac{\mathbf{a}\cdot\mathbf{b}}{2}} e^{i\mathbf{a}\cdot\mathbf{X}}\delta_n(\mathbf{x}+\hbar\mathbf{b}-\mathbf{y})$$

integrated against $\hat{f}(\mathbf{a},\mathbf{b})$. Rewriting (3) and using $\mathbf{c}=\hbar\mathbf{b}$ we get:

$$(3) = (2\pi\hbar)^{-n}\int_{\mathbb{R}^{2n}} \hat{f}\big(\mathbf{a},\frac{\mathbf{c}}{\hbar}\big) e^{i\frac{\mathbf{a}\cdot\mathbf{c}}{2}} e^{i\mathbf{a}\cdot\mathbf{X}}\delta_n(\mathbf{x}+\mathbf{c}-\mathbf{y})d\mathbf{c}d\mathbf{a} \tag{7}$$

$$= (2\pi\hbar)^{-n}\int_{\mathbb{R}^n} \hat{f}(\mathbf{a},(\mathbf{y}-\mathbf{x})/\hbar) e^{i\frac{\mathbf{a}\cdot\mathbf{y}-\mathbf{x}}{2}} e^{i\mathbf{a}\cdot\mathbf{X}}d\mathbf{a} \tag{8}$$

$$= \frac{1}{\hbar^n(2\pi)^{\frac{n}{2}}}\Big[\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\hat{f}(\mathbf{a},(\mathbf{y}-\mathbf{x})/\hbar)e^{\frac{i}{2}\mathbf{a}\cdot(\mathbf{x}+\mathbf{y})}d\mathbf{a}\Big] \tag{9}$$

The integral inside the bracket can be viewed as undoing the Fourier transform of f w.r.t. the $\mathbf{x}$ variable, leaving us with the partial transform of f in the $\mathbf{p}$ variable evaluated at the point:

$$\big(\frac{\mathbf{x}+\mathbf{y}}{2},\frac{\mathbf{y}-\mathbf{x}}{\hbar}\big)$$

Thus, the operator $Q_{weyl}(f)$ is an integral operator with kernel function, κ_f:

$$\kappa_f(\mathbf{x},\mathbf{y}) = (2\pi\hbar)^{-n}\int_{\mathbb{R}^n} f\big(\frac{\mathbf{x}+\mathbf{y}}{2},\mathbf{p}\big)e^{-\frac{i}{\hbar}(\mathbf{y}-\mathbf{x})\cdot\mathbf{p}}d\mathbf{p} \tag{10}$$

4 L^2 Theory

Let A be a self-adjoint non-negative bounded linear operator on a Hilbert space $\mathcal{H}$. Then the trace of A (w.r.t. the (e_n)-basis of $\mathcal{H}$):

$$tr(A) = \sum_{j=1}^{\infty} \langle e_j, Ae_j \rangle$$

is basis independent (which is proved in Appendix [6]). Note that this value may be infinite. Given any bounded operator, A, (not necessarily self-adjoint) we will have that A^*A is positive and self-adjoint. Then A is **Hilbert-Schmidt** if:

$$tr(A^*A) < \infty$$

It can be further shown (in 6) that for any two Hilbert-Schmidt operators, A and B, then A^*B is trace-class, (its trace is an absolutely convergent series that is independent of basis). We can then define the following inner product and norm on the space of Hilbert-Schmidt operators:

$$\langle A, B \rangle_{HS} = tr(A^*B)$$
$$\|A\|_{HS} = \sqrt{tr(A^*A)}$$

This gives us a Hilbert space on the set of Hilbert-Schmidt operators, denoted $HS(\mathcal{H})$.

We will now use the following theorem involving $L^2(\mathbb{R}^n)$.

Theorem 3. *If $\kappa \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ then for all $\psi \in L^2(\mathbb{R}^n)$ the integral,*

$$A_\kappa(\psi)(x) := \int_{\mathbb{R}^n} \kappa(x, y)\psi(y)dy \tag{11}$$

is absolutely convergent for almost all $x \in \mathbb{R}^n$, and $A_\kappa(\psi) \in L^2(\mathbb{R}^n)$. Furthermore, A_κ is Hilbert-Schmidt with norm:

$$\|A_\kappa\|_{HS} = \|\kappa\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}$$

Conversely, if A is Hilbert-Schmidt on $L^2(\mathbb{R}^n)$ then there exists unique $\kappa \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ such that $A = A_\kappa$.

This theorem is proved in *VI.23* of [2]. This theorem allows us to define the Weyl Quantization of any function in $L^2(\mathbb{R}^n)$.

Definition 1. *For all $f \in L^2(\mathbb{R}^{2n})$, define $\kappa_f : \mathbb{R}^{2n} \longrightarrow \mathbb{C}$ by:*

$$\kappa_f(\mathbf{x}, \mathbf{y}) = (2\pi\hbar)^{-n} \int_{\mathbb{R}^n} f\big(\frac{\mathbf{x}+\mathbf{y}}{2}, \mathbf{p}\big) e^{-\frac{i}{\hbar}(\mathbf{y}-\mathbf{x})\cdot\mathbf{p}} d\mathbf{p} \tag{12}$$

*and define the **Weyl Quantization** of f as an operator on $L^2(\mathbb{R}^n)$, by*

$$Q_{weyl}(f) = A_{\kappa_f}$$

where A_{κ_f} is given by (11) for the κ_f just defined.

Note that κ_f is not necessarily absolutely convergent, and should be thought of as the partial Fourier transform on the momentum variable. Theorem 3 only asserts that if κ_f is in $L^2(\mathbb{R}^{2n})$ then A_{κ_f} is absolutely convergent, in our case we may have to consider κ_f on a closed ball and then take the limit as the radius goes to infinity.

Furthermore by corollary 8.23 in Folland [[3]], we know that the Fourier transform maps the Schwartz space, $\mathcal{S}(\mathbb{R}^{2n})$, to itself. Now as $\mathcal{S}(\mathbb{R}^{2n})$ is dense in $L^2(\mathbb{R}^{2n})$ we may extend this to be a unitary operator to itself.

We now outline a method for computing κ_f at a point $(x^1, x^2) \in \mathbb{R}^{2n}$:

1. Compute the Fourier transform of $f(x, p)$, $\mathcal{F}$ on the momentum variable. This gives us $(\mathcal{F}_p f)(x, \xi)$.
2. Evaluate $(\mathcal{F}_p f)(x, \xi)$ at the point $x = \frac{x^1 - x^2}{2}$ and $\xi = \frac{x^2 - x^1}{\hbar}$.
3. Multiply the result by $(2\pi)^{-\frac{n}{2}}\hbar^{-n}$.

We thus get:

$$\kappa_f(x^1, x^2) = \frac{1}{\hbar^n (2\pi)^{\frac{n}{2}}} \big(\mathcal{F}_p f\big)\big[\frac{x^1 - x^2}{2}, \frac{x^2 - x^1}{\hbar}\big] \tag{13}$$

Putting everything together, we get the following theorem that gives us an important equivalence (although technically, a constant multiple of a unitary map):

$$L^2(\mathbb{R}^n \times \mathbb{R}^n) \cong HS(L^2(\mathbb{R}^n))$$

Theorem 4. *The operator Q_{weyl} is a constant multiple of a unitary map of $L^2(\mathbb{R}^{2n}) \longrightarrow HS(L^2(\mathbb{R}^n))$. The inverse map Q_{weyl}^{-1} is given by,*

$$Q_{weyl}^{-1}(A)(x,p) = \hbar^n \int_{\mathbb{R}^n} \kappa(x - \frac{\hbar b}{2}, x + \frac{\hbar b}{2}) e^{ib\cdot p} db \tag{14}$$

where κ is the integral kernel given to us by Theorem 3 (all such operators, A, in HS must have such unique κ).

Furthermore, for all $f \in L^2(\mathbb{R}^{2n})$ we have $Q_{weyl}(\bar{f}) = Q_{weyl}(f)^*$, thus $Q_{weyl}(f)$ is self-adjoint if f is real-valued.

Note that we should regard (14) also as an L^2 limit, as we did for (12).

Proof. Theorem (3) gives a unitary identification of $L^2(\mathbb{R}^{2n})$ with $HS(\mathbb{R}^n)$. Then since $f \mapsto \kappa_f$ is via the partial Fourier transform (which is a unitary map from $L^2(\mathbb{R}^{2n})$ to itself) we get that Q_{weyl} is a linear invertible map composed with a unitary map, hence Q_{weyl} is in fact a constant multiple of a unitary map from $L^2(\mathbb{R}^{2n}) \longrightarrow HS(L^2(\mathbb{R}^n))$.

The inverse is obtained by inverting the linear identification given by Theorem (3), and undoing the partial Fourier transform.

Finally, from our formula for κ_f in (13) it is clear that $\kappa_{\overline{f}}(x,y) = \overline{\kappa_f(y,x)}$ since letting $h(x) = \overline{f(x)}$ we know that $\widehat{h}(\xi) = \overline{\widehat{f}(-\xi)}$, hence we get (for c a real constant):

$$\begin{aligned}
\kappa_{\overline{f}}(x,y) &= c\widehat{h_p}[\frac{x-y}{2}, \frac{y-x}{\hbar}] \\
&= \overline{c\widehat{f}_p[\frac{y-x}{2}, \frac{x-y}{\hbar}]} \quad \text{(by Fourier conjugation)} \\
&= \overline{c\widehat{f}_p[\frac{y-x}{2}, \frac{x-y}{\hbar}]} \quad (c \text{ is a real constant}) \\
&= \overline{\kappa_f(y,x)}
\end{aligned}$$

Using this and the fact (proved in Appendix that integral operators, A, formed by functions from $L^2(\mathbb{R}^{2n})$ we have that A^* is also an integral operator with kernel function:

$$k^*(x,y) = \overline{k(y,x)}$$

Thus, we have $Q_{weyl}(\overline{f})$ is an integral operator with kernel $\kappa_{\overline{f}}(x,y) = \overline{\kappa_f(y,x)}$ which is the integral kernel of the adjoint of $Q_{weyl}(f)$. □

5 The Composition Formula

Recall that the product of a bounded operator and a Hilbert-Schmidt operator is again in HS. Since we have created a bijection between $L^2(\mathbb{R}^{2n})$ and $HS(L^2\mathbb{R}^n)$, given any two Weyl Quantizations there must be a unique Hilbert-Schmidt operator that is their product, we will define this to be the **Moyal product** and denote it with the "$\star$" symbol as follows:

$$Q_{weyl}(f)Q_{weyl}(g) = Q_{weyl}(f \star g) \tag{15}$$

We can describe the Moyal product via its Fourier transform:

$$\widehat{f \star g}(a,b) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{-\frac{i\hbar}{2}(a\cdot b' - b\cdot a')} \times \widehat{f}(a-a', b-b')\widehat{g}(a',b')da'db' \tag{16}$$

We immediately see that $\lim_{\hbar\to 0} f \star g = fg$ since $\hbar = 0$ shows us that the Fourier transform of $f \star g$ is just $(2\pi)^{-n}$ times the convolution of the Fourier transforms of f and g, hence it is the Fourier transform of fg. Therefore the Moyal product is a *deformation* of ordinary pointwise product of $L^2(R^{2n})$ functions. More generally, the Moyal product can be expanded in an asymptotic expansion in powers of $\hbar$, as explained in Sect. 2.3 of [4].

Proof. To prove the form in (16) we will use the form of Weyl Quantization from (3) which is shown to give the same result of our definition of Weyl Quantization given in (12) when applied to a Schwartz function (see 6). Now, we use again the exponential multiplication rule derived in 6 to get:

$$\begin{aligned}
Q_{weyl}(f)Q_{weyl}(g) &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \hat{f}(\mathbf{a},\mathbf{b})e^{i(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P})}d\mathbf{a}d\mathbf{b} \times (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \hat{g}(\mathbf{a},\mathbf{b})e^{i(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P})}d\mathbf{a}d\mathbf{b} \\
&= (2\pi)^{-2n}\int\int\int\int e^{-i\hbar(a\cdot b' - b\cdot a')/2} e^{i((a+a')\cdot X + (b+b')\cdot P)} \times \hat{f}(a,b)\hat{g}(a',b')dadbda'db'
\end{aligned}$$

Now let $c = a + a'$ and $d = b + b'$ and some minor simplification/rewriting gives us:

$$Q_{weyl}(f)Q_{weyl}(g) = (2\pi)^{-n} \int\int \left[(2\pi)^{-n} \int\int \times \hat{f}(c-a', d-b')\hat{g}(a', b')e^{-i\hbar(c\cdot b' - d\cdot a')/2}\right] e^{i(c\cdot X + d\cdot P)} da' db' dc dd$$

In this form we see that we are looking at the Weyl Quantization of the function whose Fourier transform is the function in the square brackets, which is exactly the formula (16). □

Finally, since the space of Hilbert-Schmidt operators is closed under the product operation, we have that the Moyal Product extends to a continuous binary operation on $L^2(\mathbb{R}^{2n})$ and (15) holds for all $f, g \in L^2(\mathbb{R}^{2n})$ from the standard argument of extending a map defined on a dense subset (Schwartz Space).

6 Commutation Relations

In quantum mechanics, the commutator of two operators (divided by $i\hbar$) plays a role similar to that of the Poisson bracket in classical mechanics. Thus, we may naturally ask: To what extent does the Weyl quantization (or any other quantization scheme) map Poisson brackets to commutators? The short answer is: Not always. Groenewold's No Go theorem tells us that no "reasonable" quantization scheme can give an exact correspondence be- tween $\{f, g\}$ on the classical side and $[A, B]/(i\hbar)$ on the quantum side. Nevertheless, we do see an exact correspondence for certain classes of functions.

Theorem 5. *If f is polynomial in x and p of degree at most 2, then for any polynomial g of arbitrary degree we have,*

$$Q_{weyl}(\{f, g\}) = \frac{1}{i\hbar}[Q_{weyl}(f), Q_{weyl}(g)]$$

This can be thought of as arising from the various orders of $i\hbar$ that appear in the Quantization. There is a notion of a **Moyal Bracket**:

$$\begin{aligned} \{\{f, g\}\}_{moyal} &:= \frac{1}{i\hbar}(f \star g - g \star f) \\ &= \{f, g\} + O(\hbar^2) \end{aligned} \tag{17}$$

it can be thought of as a metric of how much two observables fail to correspond with the commutator of their quantizations. It is a deformation of the classical phase-space Poisson bracket via deformation by the Planck constant, $\hbar$.

References

[1] Brian Hall. *Quantum Theory for Mathematicians*. Springer, 2013.

[2] Michael Reed and Barry Simon. *Methods of modern mathematical physics*. Acad. Press, 2007.

[3] G. B. Folland. *Real analysis : modern techniques and their applications / Gerald B. Folland.* Wiley, New York, 1984.

[4] Gerald B. Folland. *Harmonic Analysis in Phase Space. (AM-122)*. Princeton University Press, 1989.

Appendix

(A.1) The set of homogeneous polynomials, $\mathcal{P}_j$, of degree j is generated by span$\{(ax + by)^j : a, b \in \mathbb{C}\}$.

Proof. It is easy to see that there are $j + 1$ degrees of freedom in $\mathcal{P}_j$, thus $\mathcal{P}_j$ may be identified with $\mathbb{C}^{j+1}$. Now let V_j be the subspace of $\mathcal{P}_j$ created by the span of $\{(ax + by)^j : a, b \in \mathbb{C}\}$, this will be a subspace of $\mathbb{C}^{j+1}$ under our identification, and it will be closed since everything is finite dimensional. Furthermore since we are in a finite dimensional vector space we have that a subspace, $S = (S^\perp)^\perp$. Therefore, given a smooth curve $\gamma(t)$ on S we know that $\langle \gamma(t), S^\perp \rangle = 0$ hence given any $\mathbf{n} \in S^\perp$ we have that

$$\begin{aligned} 0 = \frac{d}{dt}\langle \gamma(t), \mathbf{n} \rangle &= \langle \gamma'(t), \mathbf{n} \rangle + \langle \gamma(t), 0 \rangle \\ &= \langle \gamma'(t), \mathbf{n} \rangle + 0 \\ &= \langle \gamma'(t), \mathbf{n} \rangle \end{aligned}$$

Therefore for all $t \in \mathbb{R}, \gamma'(t) \in (S^{\perp})^{\perp} = S$. So $\gamma'(t) \in S$. Applying this to our current situation we will have that smooth curves in our identification of V_j will stay within V_j. Thus, if we take the curves

$$\gamma(t) = (tx + y)^j$$

whose k-th derivatives evaluated at $t = 0$ is:

$$\gamma^{(k)}(0) = \frac{j!}{(j-k)!} x^k y^{j-k} \in V_j$$

Therefore all of the canonical homogenous polynomial basis elements of degree j are in V_j due to V_j being a linear subspace. Hence $V_j = \mathcal{P}_j$ as desired. $\square$

(A.2) Hermitian Product of Hilbert-Schmidt Operators is Trace-Class

Proof. We will show that for Hilbert-Schmidt operators A and B that:

$$\langle A, B \rangle = tr(A^* B)$$

defines an inner product that reconstruct the $\| \cdot \|_{HS}$ norm. Then by Cauchy-Schwarz we get that $tr(A^*B) < \infty$. $\square$

(A.3) Multiplication Rule for Exponentials of Operators

Proof. Let A, B be bounded linear operators on a Hilbert space that commute with their commutator. We wish to prove that:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

We will start with a few intermediate results. Namely, that for any bounded linear operators that commute, C, D we have that:

$$e^{C+D} = e^C e^D \tag{18}$$

note that this is from the fact that since the operators commute we may apply the binomial theorem to each term in the definition of the exponential of the operator $C + D$. We also need a fact that:

$$\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$$

This follows from the power series definition of the exponential of an operator and that any operator commutes with itself. Now applying this fact to:

$$\begin{aligned} \frac{d}{dt} e^{-tB} A e^{tB} &= e^{-tB} A B e^{tb} - B e^{-tB} A e^{tb} \\ &= e^{-tB} [A, B] e^{tB} \\ &= [A, B] \qquad (e^{tB} \text{ commutes with } [A, B]) \end{aligned} \tag{19}$$

We may now prove the desired statement. We will show that:

$$e^{tA} e^{tB} = e^{tA+tB+\frac{t^2}{2}[A,B]}$$

which gives us the result when $t = 1$. since $[A, B]$ commutes with both A and B by assumption, we may apply (18) to rewrite as:

$$e^{tA} e^{tB} e^{-\frac{t^2}{2}[A,B]} = e^{t(A+B)} \tag{20}$$

Let $\alpha(t)$ denote the LHS. Taking its derivative we get (and using that $[A, B]$ commutes with everything):

$$\frac{\partial \alpha}{\partial t} = e^{tA} A e^{tB} e^{-\frac{t^2}{2}[A,B]} + e^{tA} e^{tB} B e^{-\frac{t^2}{2}[A,B]} - t[A, B] e^{tA} e^{tB} e^{-\frac{t^2}{2}[A,B]}$$

Now from integrating (19) we get that:

$$e^{-tB} A e^{tB} = A + t[A, B]$$

multiplying this result by $e^{tA} e^{tB}$ we get that $e^{tA} A e^{tB} = e^{tA} e^{tB} (A + t[A, B])$. We may therefore rewrite $\frac{d}{dt}\alpha(t)$ as:

$$\begin{aligned} \frac{\partial \alpha}{\partial t} &= e^{tA} e^{tB} (A + t[A, B]) e^{-\frac{t^2}{2}[A,B]} + e^{tA} e^{tB} B e^{-\frac{t^2}{2}[A,B]} + e^{tA} e^{tB} B e^{-\frac{t^2}{2}[A,B]} (-t[A, B]) \\ &= \alpha(t) \big(A + t[A, B] + B - t[A, B] \big) \\ &= \alpha(t)(A + B) \end{aligned} \tag{21}$$

This differential equation has a unique solution given by:

$$\alpha(t) = \alpha(0) e^{t(A+B)} = I e^{t(A+B)} = e^{t(A+B)}$$

This is exactly (20), and so we are done. $\square$

(A.4) Position and Momentum Operators are Unbounded

Proof. We will start with the position operator: $X : dom(X) \longrightarrow L^2(\mathbb{R}^n)$ given by $\phi(x) \mapsto x\phi(x)$. It is true that for all $n \in \mathbb{N}$ that $\chi_{[n,n+1]} \in dom(X)$. Yet, these functions all have $\|\cdot\|_2 = 1$, while

$$\|X\chi_{[n,n+1]}\|_2 = \int_n^{n+1} x^2 dx = \frac{(n+1)^3 - n^3}{3} = n^2 + n + \frac{1}{3}$$

which goes to infinity as n does.

For the momentum operator, we instead for simplicity show that the derivative operator, D, is unbounded on its domain. Note that the function:

$$f_n(x) = \sqrt{n}e^{-n^2x^2}$$

is in L^2 since after using $u = nx\sqrt{2}$ we get:

$$\begin{aligned}\int_{\mathbb{R}} f_n(x)^2 dx &= \int_{\mathbb{R}} ne^{-2n^2x^2} dx \\ &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{-u^2} du \\ &= \sqrt{\frac{\pi}{2}}\end{aligned}$$

Now after applying D to f_n we get: $f_n'(x) = -n^2 2x f_n$, and so computing:

$$\begin{aligned}\|Df_n\|_2 &= \int_{\mathbb{R}} f_n'(x)^2 dx \\ &= \int_{\mathbb{R}} 4n^4x^2 f_n^2 dx \\ &= n^3 \int_{\mathbb{R}} x(4xn^2 e^{-2n^2x^2}) dx\end{aligned}$$

Letting $u = x$ and $v = e^{-2n^2x^2}$ we get:

$$\begin{aligned}\|Df_n\|_2 &= n^3 \Big[uv\big|_{\mathbb{R}} - \int_{\mathbb{R}} -e^{-2n^2x^2} dx \Big] \\ &= 0 + n^2 \int_{\mathbb{R}} ne^{-2n^2x^2} dx \\ &= n^2 \sqrt{\frac{\pi}{2}}\end{aligned}$$

Thus, we get that that this goes to infinity as n does, and so the derivative operator (and hence the momentum operator) is unbounded. $\square$

(A.5) Trace is Independent of Basis for Positive Linear Operators on Hilbert Spaces

Proof. It is clear that for non-negatives numbers, $(x_{j,k})$:

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} x_{j,k} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} x_{j,k} \tag{22}$$

Let A be a positive self-adjoint linear operator on a separable Hilbert space, $\mathcal{H}$, with orthonormal bases $(e_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$. Then for either basis we have by Parseval's Identity that for any $x \in \mathcal{H}$:

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|x\|^2$$

Now by definition of the trace w.r.t. the (e_n) basis we have:

$$\begin{aligned} tr_e(A) &= \sum_{n=1}^{\infty} \langle e_n, Ae_n \rangle \\ &= \sum_{n=1}^{\infty} \|\sqrt{A}e_n\|^2 \end{aligned}$$

Since A is a positive operator, it has a square root operator (which is self-adjoint as well), and so

$$\langle Av, v \rangle = \langle \sqrt{A}(\sqrt{A}v), v \rangle = \langle \sqrt{A}v, \sqrt{A}v \rangle = \|\sqrt{A}v\|^2 \tag{23}$$

Now by Parseval's Identity we get:

$$\begin{aligned} tr_e(A) &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle \sqrt{A}e_n, f_k \rangle^2 \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \langle \sqrt{A}e_n, f_k \rangle^2 \quad \text{by (22)} \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \langle e_n, \sqrt{A}f_k \rangle^2 \quad (\sqrt{A} \text{ is self-adjoint}) \\ &= \sum_{k=1}^{\infty} \|\sqrt{A}f_k\|^2 \quad \text{(Parseval's Identity)} \\ &= \sum_{k=1}^{\infty} \langle Af_k, f_k \rangle \quad \text{by (23)} \\ &= tr_f(A) \end{aligned}$$

□

(A.6) Adjoint of Integral Operators

Proof. Let T on $L^2(\mathbb{R})$ be a bounded integral operator with kernel, $\kappa(x, y)$. Then:

$$\begin{aligned} \langle Tf, g \rangle &= \int_{\mathbb{R}} (Tf)(y)\overline{g}(y)dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(x, y)f(x)\overline{g}(y)dxdy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(x, y)f(x)\overline{g}(y)dydx \quad \text{(Fubini's since } T \text{ is bounded)} \\ &= \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} \kappa(x, y)\overline{g}(y)dydx \\ &= \int_{\mathbb{R}} f(x)\langle \kappa_x, g \rangle dx \\ &= \int_{\mathbb{R}} f(x)\overline{\langle \overline{\kappa_y}, g \rangle} dx \\ &= \int_{\mathbb{R}} f(x)\overline{\int_{\mathbb{R}} \overline{\kappa}(x, y)g(y)dy}dx \end{aligned}$$

Hence, let T^* be the integral operator with kernel $\overline{\kappa}(y, x)$ (integration against x). Then we have that $\langle Tf, g \rangle = \langle f, T^*g \rangle$ for all $f, g \in L^2(\mathbb{R})$. This kernel gives us a bounded operator by Cauchy-Schwarz since T is bounded of course. □

(A.7) Schwartz Function Quantization

Proof. Let f be a Schwartz function in $L^2(\mathbb{R}^{2n})$. Then it also has a Fourier transform, $\widehat{f}$. Then we have defined

$$Q_{weyl}(f) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \widehat{f}(a, b)e^{i(a \cdot X + b \cdot P)}dadb \tag{24}$$

As X and P are operators, we may interpret this definition as a Bochner integral that maps into $\mathcal{B}(L^2(\mathbb{R}^n))$. We also know that Bochner integrals commute with linear maps. Let $\Lambda_{\phi,\psi}(A) = \langle \phi, A\psi \rangle$ be a bounded linear functional on $\mathcal{B}(L^2(\mathbb{R}^n))$. We also know that from (9) that κ_f is kernel for (23) as an integral operator and is the partial Fourier transform of f (on the momentum variable) evaluated at a translated point, denote this operation by P. Then the definition of Weyl Quantization given in definition (1) gives us:

$$\begin{aligned}
A_{\kappa_f}(\psi) &= \langle \psi, \kappa_f \rangle \\
&= \langle \psi, Pf \rangle \\
&= P\langle \psi, f \rangle \quad \text{(by Bochner Property)} \\
&= (23)
\end{aligned}$$

$\square$